

# Optimal Hölder Continuity of SHE and SWE with Rough Fractional Noise

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**Abstract:** We study the well-posedness and Hölder regularity for systems of stochastic heat equations and stochastic wave equations driven by an additive fractional Brownian sheet with temporal index  $1/2$  and spatial index  $H$  less than  $1/2$ . By using some properties of centered Gaussian field and refined estimates of Green's functions, we prove the optimal Hölder continuity for the solutions.

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## 1. Introduction and Main Results

In this paper, we consider the well-posedness and optimal Hölder regularity of the  $d$ -dimensional system of stochastic evolution equation

$$Lu_i = b_i(u) + \sum_{j=1}^d \sigma_{i,j} \frac{\partial^2 W_j}{\partial t \partial x} \quad \text{in } \mathbb{R}_+ \times \mathbb{R}, \quad (1.1)$$

with  $i \in \mathbb{N}_d := \{1, \dots, d\}$ . Here  $L$  is a second order differential operator,  $b = (b_i)_{i \in \mathbb{N}_d}$  is an  $\mathbb{R}^d$ -valued function,  $\sigma = (\sigma_{i,j})_{i,j \in \mathbb{N}_d}$  is an  $\mathbb{R}^{d \times d}$ -valued constant matrix and  $W = (W_1, \dots, W_d)$  is an  $\mathbb{R}^d$ -valued fractional Brownian sheet (fBs) with temporal index  $1/2$  and spatial index  $H \in (0, 1/2)$  on the stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ , i.e., for any  $(t, x), (s, y) \in \mathbb{R}_+ \times \mathbb{R}$  and  $i, j \in \mathbb{N}_d$ ,

$$\mathbb{E}[W_i(t, x)W_j(s, y)] = \delta_{ij}(t \wedge s) \frac{|x|^{2H} + |y|^{2H} - |x - y|^{2H}}{2}. \quad (1.2)$$

We mainly focus on the stochastic heat equation (SHE), i.e.,  $L = \partial_t - \partial_{xx}$ , and the stochastic wave equation (SWE), i.e.,  $L = \partial_{tt} - \partial_{xx}$ , with specified initial data. In the SHE case we impose  $u(0, x) = u_0(x)$ , while in the SWE case we further impose  $u_t(0, x) = v_0(x)$ .

There has been a widespread interest in Hölder continuity result for random fields. This type of sample path regularity is a key property that is needed early

in any fine study of a random field. For instance, Hölder exponents are useful to analyze probabilistic potential theory (see e.g. [8, 9, 10, 13] and references therein). More precisely, a upper bound of hitting probability for a random field follows from the Hölder continuity combining the existence of a density which is uniformly bounded (see e.g. [8], Theorem 3.3 or [9], Theorem 2.1). Optimal Hölder exponents are also used to verify the optimality of convergence rate of numerical schemes (see e.g. [5]).

Given constants  $\beta_1, \beta_2 \in (0, 1]$ , denote by  $\mathcal{C}_{\beta_1, \beta_2} := \mathcal{C}_{\beta_1, \beta_2}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}^d)$  the set of functions  $v : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}^d$  which is temporally  $\beta_1$ -Hölder continuous and spatially  $\beta_2$ -Hölder continuous. More precisely, for each compact subset  $D \subset \mathbb{R}_+ \times \mathbb{R}$ , there is a finite constant  $C$  such that for all  $(t, x), (s, y) \in D$ ,

$$\|v(t, x) - v(s, y)\| \leq C(|t - s|^{\beta_1} + |x - y|^{\beta_2}),$$

where we denote  $\|v\| := \sqrt{v_1^2 + \cdots + v_d^2}$  for  $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ . Let

$$\mathcal{C}_{\beta_1-, \beta_2-} := \bigcap_{0 < \alpha_1 < \beta_1} \bigcap_{0 < \alpha_2 < \beta_2} \mathcal{C}_{\alpha_1, \alpha_2} \quad \text{and} \quad \mathcal{C}_{\beta_1+, \beta_2+} := \bigcap_{\beta_1 < \alpha_1 \leq 1} \bigcap_{\beta_2 < \alpha_2 \leq 1} \mathcal{C}_{\alpha_1, \alpha_2}.$$

Similarly, one can also define  $\mathcal{C}_{\beta_1}(\mathbb{R}_+)$ ,  $\mathcal{C}_{\beta_1 \pm}(\mathbb{R}_+)$  and, respectively,  $\mathcal{C}_{\beta_2}(\mathbb{R})$ ,  $\mathcal{C}_{\beta_2 \pm}(\mathbb{R})$  as the Hölder space of  $v$  in temporal direction and spatial direction.

Many authors consider the sample path Hölder continuity for the solutions of SHE and SWE driven by temporally white and spatially homogeneous colored noise; see e.g. [7, 11, 12, 16, 17, 18, 20] and references therein. In the case of SHE over  $\mathbb{R}^d$ , assuming that the spectral measure  $\tilde{\mu}$  of the noise satisfies

$$\int_{\mathbb{R}^d} \frac{\tilde{\mu}(d\xi)}{(1 + |\xi|^2)^\eta} < \infty \quad \text{for some } \eta \in (0, 1), \quad (1.3)$$

and if the initial data is a bounded  $\rho$ -Hölder continuous function for some  $\rho \in (0, 1)$ , the authors in [20], Theorem 2.3, prove that

$$u \in \mathcal{C}_{\frac{\rho \wedge (1-\eta)}{2}-, (\rho \wedge (1-\eta))-(\mathbb{R}_+ \times \mathbb{R}^d)}, \quad \text{a.s.},$$

where  $a \wedge b := \min\{a, b\}$ . Based on the fractional Sobolev imbedding theorem combining the Fourier transformation technique, [7, 11, 12, 16] generalize to the case of SWE with  $d = 1, 2, 3$  driven by similar noise whose spectral measure is given by a Riesz kernel, i.e.,  $\Gamma(dx) = |x|^{-\alpha}$ ,  $\alpha \in (0, d)$ , or equivalently,  $\tilde{\mu}(dx) = C|x|^{\alpha-d}$  for some constant  $C$ . In this special case, (1.3) holds if and only if  $\alpha \in (0, (2\eta) \wedge d)$ . It is shown that the corresponding solution of SWE

$$u \in \mathcal{C}_{\rho \wedge (1-\eta)-, (\rho \wedge (1-\eta))-(\mathbb{R}_+ \times \mathbb{R}^d)}, \quad \text{a.s.}$$

For the case of space-time white noise on  $\mathbb{R}_+ \times \mathbb{R}$ , the spectral measure  $\tilde{\mu}$  is Lebesgue measure and hence the exponent  $\eta$  in (1.3) (with  $d = 1$ ) can take the value  $\frac{1}{2} + \epsilon$  for any  $\epsilon > 0$ . Their results implies that

$$u \in \mathcal{C}_{\frac{1}{4}-, \frac{1}{2}-}(\mathbb{R}_+ \times \mathbb{R}) \text{ a.s. for SHE} \quad \text{and} \quad u \in \mathcal{C}_{\frac{1}{2}-, \frac{1}{2}-}(\mathbb{R}_+ \times \mathbb{R}) \text{ a.s. for SWE.}$$

See also [21] for SHE and SWE over bounded interval driven by time-space white noise with homogeneous Dirichlet or Neumann boundary condition. We also remark that [18] and, respectively, [17], applying Feymann-Kac formula, derive the Hölder regularity of SHE driven by fBs with each component index  $H \in (1/2, 1)$  and by temporal fractional with  $H \in (0, 1/2)$  and spatial homogeneous smooth noise.

All the above mentioned papers consider noises whose spectral measure, in the spatial direction, are non-negative and locally integrable. Unfortunately, the spectral measure  $\mu$  of the fractional noise with index  $H \in (0, 1/2)$  defined by

$$\mu(d\xi) = c_H |\xi|^{1-2H} d\xi, \quad c_H = \frac{\Gamma(2H+1) \sin(\pi H)}{2\pi}, \quad (1.4)$$

where  $\Gamma$  is the Gamma function, is not locally integrable. This motivates us the present paper, in which we investigate the Hölder regularity of the temporally white and spatially fractional with index  $H \in (0, 1/2)$  driven Eq. (1.1).

We need the following assumptions on the drift coefficient  $b$  and the initial data  $u_0$  and  $v_0$ . For  $v = (v_1, \dots, v_d) \in \mathbb{R}^d$  and  $p \in (0, \infty)$ , we denote

$$\|v\|_{L^p} := \left( \mathbb{E} \left[ (v_1^2 + \dots + v_d^2)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}}.$$

**Assumption 1.1.**  $b$  is Lipschitz continuous, i.e.,

$$L_b := \sup_{u_1 \neq u_2} \frac{\|b(u_1) - b(u_2)\|}{\|u_1 - u_2\|} < \infty. \quad (1.5)$$

**Assumption 1.2.**  $u_0$  and  $v_0$  are stochastically  $\alpha$ -Hölder continuous with  $\alpha \in (0, 1]$ , i.e., for all  $p \geq 1$ , there exists  $L_0 = L_0(p) \in (0, \infty)$  such that

$$\|u_0(x) - u_0(y)\|_{L^p} + \|v_0(x) - v_0(y)\|_{L^p} \leq L_0 |x - y|^\alpha, \quad x, y \in \mathbb{R}. \quad (1.6)$$

Throughout the paper, we consider the following parabolic and hyperbolic metric. For all  $(t, x), (s, y) \in [0, T] \times \mathbb{R}$ , we denote

$$\Delta((t, x); (s, y)) := \begin{cases} |t - s|^{\frac{1}{2}} + |x - y|, & \text{for SHE;} \\ |t - s| + |x - y|, & \text{for SWE.} \end{cases}$$

Our main result is the following well-posedness and optimal Hölder regularity results for Eq. (1.1).

**Theorem 1.1.** *Let Assumption 1.1 hold. Assume that  $u_0$  and  $v_0$  are continuous and bounded and possess bounded  $p$ -th moment for  $p \geq 2$ . Then for any  $T \in (0, \infty)$ , Eq. (1.1) has a unique mild solution  $u = \{u(t, x) : (t, x) \in [0, T] \times \mathbb{R}\}$  which is an adapted process satisfying  $\sup_{(t, x) \in [0, T] \times \mathbb{R}} \mathbb{E}[\|u(t, x)\|^p] < \infty$ ,  $p \geq 2$ . If, in addition, Assumption 1.2 holds, then there exist  $C = C(p, T, H, d)$  such that for any  $(t, x), (s, y) \in [0, T] \times \mathbb{R}$ ,*

$$\|u(t, x) - u(s, y)\|_{L^p} \leq C (\Delta((t, x); (s, y)))^{\alpha \wedge H}. \quad (1.7)$$

Moreover, this estimate is optimal in the sense that the reverse estimate of (1.7) holds with  $\alpha \wedge H$  replaced by  $H$ , in compact intervals with  $s$  and  $t$  being sufficiently close, when  $u_0 = v_0 = 0$ ,  $b = 0$  and  $\sigma = I_{d \times d}$ . As a result,  $u$  has a version which is in  $\mathcal{C}_{\frac{\alpha \wedge H}{2}-, (\alpha \wedge H)-}$ , but any version of  $u$  is not in  $\mathcal{C}_{\frac{H}{2}+, H+}$ .

We remark that, due to the complex spatial structure of the fBs  $W$  determined by (1.2), the well-posedness of Eq. (1.1) with general Lipschitz continuous diffusion coefficient  $\sigma(u)$  is an open problem. Recently, [3, 4] have established the existence of a unique mild solution and its Hölder continuity of Eq. (1.1) with vanishing drift and affine diffusion, i.e.,  $\sigma(u) = a_1 u + a_2$  with  $a_1, a_2 \in \mathbb{R}$ . For special nonlinear diffusion  $\sigma(u)$  which is differentiable with a Lipschitz derivative and satisfies  $\sigma(0) = 0$ , [15] obtains similar well-posed result. In these multiplicative case, the Hurst index is restricted as  $H \in (1/4, 1/2)$  because of technical requirements. In our additive case,  $H$  is allowed to be less than  $1/4$ . Moreover, our estimate for moments' increments is valid uniformly with respect to the time-space parameter, while that of [3] holds for these parameter close enough from which they obtain locally Hölder continuity of the solution; see Theorem 1 in [3] and Remark 3.1 below. We recall that [5] has investigated the Sobolev regularity of the solution of Eq. (1.1) and Wong-Zakai approximations for the proposed noise to numerically solve Eq. (1.1) with homogeneous Dirichlet or Neumann boundary condition.

Some preliminaries including required estimates about Green's functions and stochastic integral with respect to the fBs  $W$  are given in the next Section. In Section 3, we prove the well-posedness and Hölder continuity of Eq. (1.1). Finally, in the last Section, we prove the optimality of Hölder exponents established in Section 3.

We end this Section by introducing the following frequently used notations:

1.  $T \in (0, \infty)$  is fixed throughout the paper.
2. Without illustrated, all supremum with respect to  $t$  (respectively,  $x$  and  $n$ ) denotes  $\sup_{t \in [0, T]}$  (respectively,  $\sup_{x \in \mathbb{R}}$  and  $\sup_{n \in \mathbb{N}_+}$ ).
3.  $C$  (and  $C_1, C_2$  etc.) will denote a generic constant which may change from line to line.

## 2. Preliminaries

### 2.1. Green's Functions and Mild Solutions

We denote by  $G_t(x)$  the Green's function of  $Lu(t, x) = 0$ . It has explicit form

$$G_t(x) = \begin{cases} \frac{1}{\sqrt{4\pi t}} \exp(-\frac{|x|^2}{4t}), & \text{for SHE;} \\ \frac{1}{2} \mathbf{1}_{|x| < t}, & \text{for SWE,} \end{cases} \quad (2.1)$$

where  $\mathbf{1}$  denotes the indicator function. The mild solution of Eq. (1.1) is defined as an  $\mathcal{F}_t$ -adapted random field satisfying

$$u(t, x) = \omega(t, x) + \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x-\eta) b(u(\theta, \eta)) d\theta d\eta + \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x-\eta) \sigma W(d\theta, d\eta). \quad (2.2)$$

Here  $\omega(t, x)$  is the solution of the homogeneous equation with the same initial conditions as given in Section 1. More precisely,

$$\omega(t, x) = \begin{cases} \frac{1}{\sqrt{4\pi t}} \int_0^t \exp(-\frac{|x-\eta|^2}{4t}) u_0(\eta) d\eta, & \text{for SHE;} \\ \frac{1}{2} (u_0(x+t) - u_0(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} v_0(\eta) d\eta, & \text{for SWE.} \end{cases}$$

We need the following known results.

**Lemma 2.1.** 1. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a tempered function whose Fourier transform in  $S'(\mathbb{R})$  is a locally integrable function. Then for any  $H \in (0, 1/2)$ ,

$$c_H \int_{\mathbb{R}} |\mathcal{F}g(\xi)|^2 |\xi|^{1-2H} d\xi = C_H \int_{\mathbb{R}^2} \frac{|g(x) - g(y)|^2}{|x - y|^{2-2H}} dx dy \quad (2.3)$$

when either one of the two integrals is finite, with  $c_H$  given by (1.4) and  $C_H = \frac{H(1-2H)}{2}$ .

2. The integral  $\int_{\mathbb{R}} |\mathcal{F}G(\xi)|^2 |\xi|^\alpha d\xi$  converges if and only if  $\alpha \in (-1, 1)$ . Moreover, when it converges, there exist  $C_1 = C_1(\alpha)$  and  $C_2 = C_2(\alpha)$  such that for any  $t \geq 0$ ,

$$\int_0^t \int_{\mathbb{R}} |\mathcal{F}G_\theta(\xi)|^2 |\xi|^\alpha d\theta d\xi = \begin{cases} C_1 t^{\frac{1-\alpha}{2}}, & \text{for SHE;} \\ C_2 t^{2-\alpha}, & \text{for SWE.} \end{cases} \quad (2.4)$$

3. For any  $\alpha \in (-1, 1)$  and any  $h \in \mathbb{R}$ , there exist  $C = C(T)$  such that

$$\int_0^T \int_{\mathbb{R}} |\mathcal{F}G_{t+h}(\xi) - \mathcal{F}G_t(\xi)|^2 |\xi|^\alpha d\xi dt \leq \begin{cases} Ch^{\frac{1-\alpha}{2}}, & \text{for SHE;} \\ Ch^{1-\alpha}, & \text{for SWE.} \end{cases} \quad (2.5)$$

$$\int_0^T \int_{\mathbb{R}} (1 - \cos(\xi h)) |\mathcal{F}G_t(\xi)|^2 |\xi|^\alpha dy dt \leq C|h|^{1-\alpha}. \quad (2.6)$$

*Proof.* We refer to [4], Proposition 2.8, Lemma 3.1, Lemma 3.5, and Lemma 3.4 for (2.3), (2.4), (2.5) and (2.6), respectively.  $\square$

## 2.2. Fractional Noise and Stochastic Calculus

In this subsection, we define the stochastic integral with respect to fBs  $W$  determined by (1.2) with  $H \in (0, 1/2)$  in one dimensional case, i.e.,  $d = 1$ . The arguments can be straightforward extended to general dimension  $d \in \mathbb{N}_+$ .

Recall that the fractional Brownian motion (fBm) in  $\mathbb{R}$  with index  $H \in (0, 1)$  is a centered Gaussian process  $B = \{B(x)\}_{x \in \mathbb{R}}$  with covariance

$$\mathbb{E}[B(x)B(y)] = \int_{\mathbb{R}} \mathcal{F}\mathbf{1}_{[0,x]}(\xi) \overline{\mathcal{F}\mathbf{1}_{[0,y]}(\xi)} \mu(d\xi).$$

In particular, the fBm with index  $H = 1/2$  coincides with the Brownian motion. It is straightforward to verify that when  $H \in (0, 1/2)$  then for any  $\eta \in (1-H, \infty)$ ,

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^\eta} < \infty.$$

On the other hand, when  $H \in (1/2, 1)$ , the Fourier transform of  $\mu$  is the locally integrable function  $f(x) = H(2H - 1)|x|^{2H-2}$ . In this case, it is orthogonal to the Gaussian random field determined by a Riesz kernel, which is investigated by a lot of authors (see e.g. [6, 7, 10, 11, 12, 13, 16, 20] and references therein). However, when  $H \in (0, 1/2)$ , the Fourier transform of  $\mu$  is a genuine distribution and not a function. Therefore, the technicals used in the aforementioned references cannot be applied to the case of  $H \in (0, 1/2)$ .

The author in [19] has shown that the domain of the Wiener integral with respect to the fBm  $B$  in  $\mathbb{R}$  with index  $H \in (0, 1)$  is the completion of  $\mathcal{C}_0^\infty(\mathbb{R})$  with respect to the inner product

$$\langle \varphi, \psi \rangle := \mathbb{E}[B(\varphi)B(\psi)] = \int_{\mathbb{R}} \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)} \mu(d\xi), \quad \varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}), \quad (2.7)$$

which coincides with the space of distribution  $S \in \mathcal{S}'(\mathbb{R})$ , whose Fourier transform  $\mathcal{F}S$  is a locally integrable function which satisfies  $\int_{\mathbb{R}} |\mathcal{F}S(\xi)|^2 \mu(d\xi) < \infty$ . There is a Hilbert space  $\mathcal{H}^B$  naturally associated with the fBm  $B$ . Indeed, let  $\mathcal{H}^B$  be the completion of  $\mathcal{C}_0^\infty(\mathbb{R})$ , the space of infinitely differentiable functions with compact support, with respect to the inner product defined by (2.7). Therefore,  $\mathcal{H}^B$  is the reproducing kernel Hilbert space (RKHS) of the fBm  $B$ .

In this level,  $W$  is a spatially homogeneous Gaussian random field, which is described as follows:  $W$  is a centered Gaussian family of random variables  $\{W(\varphi) : \varphi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R})\}$  defined in a complete filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ , with covariance

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_0^T \int_{\mathbb{R}} \mathcal{F}\varphi(t, \cdot)(\xi) \overline{\mathcal{F}\psi(t, \cdot)(\xi)} \mu(d\xi) dt =: \langle \varphi, \psi \rangle_{\mathcal{H}_T} \quad (2.8)$$

for any  $\varphi, \psi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R})$ , where  $\mu$  is defined by (1.4). Here  $\mathcal{H}_T$ , the completion of  $\mathcal{C}_0^\infty([0, T] \times \mathbb{R})$  with respect to the inner product defined by (2.8), is the RKHS of the fBs  $W$ . It can be identified with the homogenous Sobolev space of order  $1/2 - H$  of functions with values in  $L^2(\mathbb{R})$  (see e.g. [2, 14, 15]). Nevertheless, we will use another simpler characterization of  $\mathcal{H}_T$ , in terms of (2.3) in Lemma 2.1, which is more suitable in our case:

$$\mathcal{H}_T := \left\{ \phi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}) : \int_0^T \int_{\mathbb{R}^2} \frac{|\phi(s, y) - \phi(s, z)|^2}{|y - z|^{2-2H}} dy dz ds < \infty \right\}.$$

We have the following Itô isometry.

**Theorem 2.1.** *Assume that  $\varphi \in \mathcal{H}_T$ . Then for any  $t \in [0, T]$ ,*

$$\mathbb{E} \left[ \left| \int_0^t \int_{\mathbb{R}} \varphi(s, y) W(d\theta, d\eta) \right|^2 \right] = \int_0^t \int_{\mathbb{R}} |\mathcal{F}\varphi(s, \cdot)(\xi)|^2 \mu(d\xi) ds. \quad (2.9)$$

*Proof.* See [4], Theorem 2.7 or [15], Proposition 2.3 or [19], Proposition 4.1.  $\square$

**Remark 2.1.** 1. When  $H = 1/2$ , then the noise reduces to time-space white noise:  $\mu(d\xi) = d\xi$ . By Plancherel theorem, we have

$$\mathbb{E} \left[ \left| \int_0^t \int_{\mathbb{R}} \varphi(s, y) W(d\theta, d\eta) \right|^2 \right] = \int_0^t \int_{\mathbb{R}} |\varphi(s, y)|^2 dy ds.$$

Thus all main results, such as Theorem 1.1, of the paper hold for  $H = 1/2$ .  
 2. Since  $\int_0^t \int_{\mathbb{R}} \varphi(s, y) W(d\theta, d\eta)$  is a centered Gaussian process, for any  $p \geq 2$ , there exists  $C = C(p)$  such that for any  $t \in [0, T]$ ,

$$\mathbb{E} \left[ \left| \int_0^t \int_{\mathbb{R}} \varphi(s, y) W(d\theta, d\eta) \right|^p \right] = C \left( \int_0^t \int_{\mathbb{R}} |\mathcal{F}\varphi(s, \cdot)(\xi)|^2 \mu(d\xi) ds \right)^{\frac{p}{2}}. \quad (2.10)$$

### 3. Well-posedness and Hölder Continuity

We use the Picard iteration scheme defined below to prove the well-posedness of Eq. (1.1) in this Section, and then derive the sample path Hölder continuity of the solution.

Define the Picard iteration scheme as

$$\begin{aligned} u^0(t, x) &:= \omega(t, x); \\ u^{n+1}(t, x) &:= \omega(t, x) + \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x - \eta) b(u^n(\theta, \eta)) d\theta d\eta \\ &\quad + \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x - \eta) \sigma W(d\theta, d\eta), \quad n = 0, 1, 2, \dots \end{aligned} \quad (3.1)$$

**Theorem 3.1.** Let Assumption 1.1 hold. Assume that  $u_0$  and  $v_0$  are continuous and bounded and possess bounded  $p$ -th moment for  $p \geq 2$ . Then Eq. (1.1) has a unique mild solution  $u = \{u(t, x) : (t, x) \in [0, T] \times \mathbb{R}\}$  which is an adapted process satisfying  $\sup_{t,x} \mathbb{E}[|u(t, x)|^p] < \infty$ ,  $p \geq 2$ . If, in addition, Assumption 1.2 holds, then for any  $p \geq 2$  and any  $(t, x), (s, y) \in [0, T] \times \mathbb{R}$ ,

$$\|u(t, x) - u(s, y)\|_{L^p} \leq C (\triangle((t, x); (s, y)))^{\alpha \wedge H}. \quad (3.2)$$

As a result,  $u$  has a version which is in  $\mathcal{C}_{\frac{\alpha \wedge H}{2}-, (\alpha \wedge H)-}$ .

*Proof.* We start with verifying the uniform boundedness of the  $p$ -th moments of  $u^n$ . The Hölder inequality and equality (2.10) implies the existence of  $C = C(p, H, T, d)$  such that

$$\begin{aligned} \mathbb{E}[\|u^{n+1}(t, x)\|^p] &\leq C \mathbb{E}[\|\omega(t, x)\|^p] + C \int_0^t \mathbb{E}[\|\int_{\mathbb{R}} G_{t-\theta}(x - \eta) b(u^n(\theta, \eta)) dy\|^p] ds \\ &\quad + C \left( \int_0^t \int_{\mathbb{R}} |\mathcal{F}G_{\theta}(x - \cdot)(\xi)|^2 |\xi|^{1-2H} d\theta d\xi \right)^{\frac{p}{2}}. \end{aligned}$$

Since  $u_0$  and  $v_0$  are continuous and bounded and possess bounded  $p$ -th moment,  $\mathbb{E}[\|\omega(t, x)\|^p]$  is bounded uniformly. Applying the property of Fourier transform:  $\mathcal{F}g(x + \cdot)(\xi) = e^{ix\xi} \mathcal{F}g(\xi)$ ,  $\xi$ -a.e., and the estimate (2.4), the last term is bounded uniformly. By Hölder inequality,

$$\begin{aligned} & \int_0^t \mathbb{E} \left[ \left\| \int_{\mathbb{R}} G_{t-\theta}(x - \eta) b(u^n(\theta, \eta)) dy \right\|^p \right] ds \\ & \leq \int_0^t \left( \int_{\mathbb{R}} G_{t-\theta}(x - \eta) dy \right)^p \left( \sup_{x \in \mathbb{R}} \mathbb{E}[\|b(u^n(\theta, x))\|^p] \right) d\theta \\ & \leq C + C \int_0^t \left( \sup_{x \in \mathbb{R}} \mathbb{E}[\|u^n(\theta, x)\|^p] \right) g(t - \theta) d\theta, \end{aligned}$$

where

$$g(t) := \left( \int_{\mathbb{R}} G_t(y) dy \right)^p = \begin{cases} 1, & \text{for SHE} \\ (2t)^p, & \text{for SWE} \end{cases}$$

is uniformly bounded in  $[0, T]$ . It follows that

$$\mathbb{E}\|u^{n+1}(t, x)\|^p \leq C + C \int_0^t \left( \sup_{x \in \mathbb{R}} \mathbb{E}[\|u^n(\theta, x)\|^p] \right) d\theta.$$

Therefore, if we set  $M^n(t) := \sup_{x \in \mathbb{R}} \mathbb{E}|u^n(t, x)|^p$ , then

$$M^{n+1}(t) \leq C + C \int_0^t M^n(\theta) d\theta.$$

Gronwall Lemma yields  $\sum_{n=1}^{\infty} M^n(t)$  converges uniformly on  $[0, T]$ . In particular,  $\sup_n \sup_t M^n(t) < \infty$ , i.e.,  $\sup_n \sup_{t,x} \mathbb{E}[|u^n(t, x)|^p] < \infty$ .

Next we prove that the scheme (3.1) is convergent and the limit is the unique mild solution of Eq. (1.1). It is clear that

$$u^{n+1}(t, x) - u^n(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x - \eta) (b(u^n(\theta, \eta)) - b(u^{n-1}(\theta, \eta))) d\theta d\eta.$$

Similarly to the proof of moments' boundedness of  $u^n$ , we have

$$\mathbb{E}\|u^{n+1}(t, x) - u^n(t, x)\|^p \leq C \int_0^t \left( \sup_{x \in \mathbb{R}} \mathbb{E}\|u^n(\theta, x) - u^{n-1}(\theta, x)\|^p \right) d\theta,$$

Define  $H^n(t) := \sup_{x \in \mathbb{R}} \mathbb{E}|u^{n+1}(t, x) - u^n(t, x)|^p$ . Then

$$H^n(t) \leq C \int_0^t H^{n-1}(\theta) d\theta,$$

from which we conclude by Gronwall Lemma that  $\sum_{n=1}^{\infty} H^n(t)$  converges uniformly on  $[0, T]$ . This shows that for each  $t$  and  $x$ ,  $u^n(t, x)$  converges in  $L^p(\mathbb{P})$



to  $u(t, x)$ , which is the mild solution of (1.1) satisfying  $\sup_{t,x} \mathbb{E}[|u(t, x)|^p] < \infty$ . The same procedure yields the uniqueness of the mild solution of (1.1).

Finally we prove the Hölder continuity of the solution of Eq. (1.1). Without loss of generality, assume that  $0 \leq s < t \leq T$ . For SHE, applying the semigroup property of  $G$  as well as the fact that  $\int_{\mathbb{R}} G_t(\eta) d\eta = 1$  and then using Hölder inequality, we obtain

$$\begin{aligned} & \mathbb{E} \left[ \left\| \int_{\mathbb{R}} (G_t(x - \eta) - G_s(x - \eta)) u_0(\eta) d\eta \right\|^p \right] \\ &= \mathbb{E} \left[ \left\| \int_{\mathbb{R}} G_{t-s}(\eta) \left( \int_{\mathbb{R}} G_s(x - z) (u_0(z - \eta) - u_0(z)) dz \right) d\eta \right\|^p \right] \\ &\leq \int_{\mathbb{R}} G_{t-s}(\eta) \int_{\mathbb{R}} G_s(x - z) \mathbb{E}[|u_0(z - \eta) - u_0(z)|^p] dz d\eta \\ &\leq \int_{\mathbb{R}} G_{t-s}(\eta) |\eta|^{p\alpha} d\eta = \frac{2^{\frac{m}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{p\alpha + 1}{2}\right) |t - s|^{\frac{p\alpha}{2}}. \end{aligned}$$

Triangle inequality and Assumption 1.2 then yield

$$\begin{aligned} & \mathbb{E}[\|\omega(t, x) - \omega(s, y)\|^p] \\ &\leq C \mathbb{E}[\|\int_{\mathbb{R}} (G_t(x - \eta) - G_s(x - \eta)) u_0(\eta) d\eta\|^p] + C \mathbb{E}[\|\int_{\mathbb{R}} (G_s(x - \eta) - G_s(y - \eta)) u_0(\eta) d\eta\|^p] \\ &\leq C |t - s|^{\frac{p\alpha}{2}} + C \left( \int_{\mathbb{R}} G_s(\eta) \mathbb{E}[\|u_0(x + \eta) - u_0(y + \eta)\|^p] d\eta \right) \\ &\leq C \left( |t - s|^{\frac{p\alpha}{2}} + |x - y|^{p\alpha} \right). \end{aligned}$$

For SWE, by Assumption 1.2 and the boundedness of  $v_0$ , we have after applying Young inequality that

$$\begin{aligned} & \mathbb{E}[\|\omega(t, x) - \omega(s, y)\|^p] \\ &\leq C \mathbb{E}[\|u_0(x + t) + u_0(x - t) - u_0(y + s) - u_0(y - s)\|^p] \\ &\quad + C \mathbb{E}[\|\int_{\mathbb{R}} (G_t(x - \eta) - G_s(x - \eta)) v_0(\eta) d\eta\|^p] + C \mathbb{E}[\|\int_{\mathbb{R}} (G_s(x - \eta) - G_s(y - \eta)) v_0(\eta) d\eta\|^p] \\ &\leq C (|t - s|^{p\alpha} + |x - y|^{p\alpha}). \end{aligned}$$

Therefore, we have proved

$$\mathbb{E}[\|\omega(t, x) - \omega(s, y)\|^p] \leq C (\Delta((t, x); (s, y)))^{p\alpha}.$$

Applying equality (2.10) and Itô isometry (2.9), we obtain

$$\begin{aligned} & \mathbb{E} \left[ \left\| \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x - \eta) \sigma W(d\theta, d\eta) - \int_0^s \int_{\mathbb{R}} G_{s-\theta}(y - \eta) \sigma W(d\theta, d\eta) \right\|^p \right] \\ &\leq C \left( \int_s^t \int_{\mathbb{R}} |\mathcal{F} G_{t-\theta}(\xi)|^2 |\xi|^{1-2H} d\theta d\xi \right)^{\frac{p}{2}} + C \left( \int_0^s \int_{\mathbb{R}} |\mathcal{F} (G_{t-\theta}(x - \cdot) - G_{s-\theta}(x - \cdot))(\xi)|^2 |\xi|^{1-2H} d\theta d\xi \right)^{\frac{p}{2}} \\ &\quad + C \left( \int_0^s \int_{\mathbb{R}} |\mathcal{F} (G_{s-\theta}(x - \cdot) - G_{s-\theta}(y - \cdot))(\xi)|^2 |\xi|^{1-2H} d\theta d\xi \right)^{\frac{p}{2}}. \end{aligned}$$

The fact that  $\mathcal{F}g(x + \cdot)(\xi) = e^{ix\xi}\mathcal{F}g(\xi)$ ,  $\xi$ -a.e., and the estimates (2.4)–(2.6) imply that the above three terms can be controlled by

$$\begin{aligned} & C\left(\int_0^{t-s} \int_{\mathbb{R}} |\mathcal{F}G_\theta(\xi)|^2 |\xi|^{1-2H} d\theta d\xi\right)^{\frac{p}{2}} + C\left(\int_0^s \int_{\mathbb{R}} |\mathcal{F}G_{t-s+\theta}(\xi) - \mathcal{F}G_\theta(\xi)|^2 |\xi|^{1-2H} d\theta d\xi\right)^{\frac{p}{2}} \\ & + C\left(\int_0^s \int_{\mathbb{R}} (1 - \cos(\xi(x-y))) |\mathcal{F}G_\theta(\xi)|^2 |\xi|^{1-2H} d\theta d\xi\right)^{\frac{p}{2}} \leq C(\Delta((t,x);(s,y)))^{Hp}. \end{aligned}$$

By Young inequality and change of variables, we have

$$\begin{aligned} & \mathbb{E} \left[ \left\| \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x-\eta) b(u(\theta, \eta)) d\theta d\eta - \int_0^s \int_{\mathbb{R}} G_{s-\theta}(y-\eta) b(u(\theta, \eta)) d\theta d\eta \right\|^p \right] \\ & \leq C \mathbb{E} \left[ \left\| \int_0^{t-s} \int_{\mathbb{R}} G_{t-\theta}(x-\eta) b(u(\theta, \eta)) d\theta d\eta \right\|^p \right] \\ & \quad + C \mathbb{E} \left[ \left\| \int_{t-s}^t \int_{\mathbb{R}} G_{t-\theta}(x-\eta) (b(u(\theta, \eta)) - b(u(\theta - (t-s), \eta - (x-y)))) d\theta d\eta \right\|^p \right]. \end{aligned}$$

Since  $b$  is Lipschitz continuous and  $\mathbb{E}[|u(t, x)|^p]$  is uniformly bounded,

$$\mathbb{E} \left[ \left\| \int_0^{t-s} \int_{\mathbb{R}} G_{t-\theta}(x-\eta) b(u(\theta, \eta)) d\theta d\eta \right\|^p \right] \leq \begin{cases} C|t-s|^p, & \text{for SHE;} \\ C|t-s|^{2p}, & \text{for SWE.} \end{cases}$$

Applying Hölder inequality and change of variables, we obtain

$$\begin{aligned} & \mathbb{E} \left[ \left\| \int_{t-s}^t \int_{\mathbb{R}} G_{t-\theta}(x-\eta) (b(u(\theta, \eta)) - b(u(\theta - (t-s), \eta - (x-y)))) d\theta d\eta \right\|^p \right] \\ & \leq C \int_0^s \sup_{\eta \in \mathbb{R}} \mathbb{E} [\|u(\theta + (t-s), \eta) - u(\theta, \eta - (x-y))\|^p] ds. \end{aligned}$$

The above estimates and Young inequality yield

$$\mathbb{E}[\|u(t, x) - u(s, y)\|^p] \leq C(\Delta((t, x);(s, y)))^{p(\alpha \wedge H)} + C \int_0^s \sup_{\eta \in \mathbb{R}} \mathbb{E}[\|u(\theta + (t-s), \eta) - u(\theta, \eta - (x-y))\|^p] ds.$$

We conclude by Gronwall inequality that

$$\mathbb{E}[\|u(t, x) - u(s, y)\|^p] \leq C(\Delta((t, x);(s, y)))^{p(\alpha \wedge H)}.$$

By Kolmogorov continuity theorem,  $u$  has a version in  $\mathcal{C}_{\frac{\alpha \wedge H}{2}, (\alpha \wedge H)-}$ .  $\square$

**Remark 3.1.** For Eq. (1.1) with  $d = 1$  and vanishing drift driven by affine noise, i.e.,  $\sigma = \sigma_1 u + \sigma_2$  with  $\sigma_1, \sigma_2 \in \mathbb{R}$ , the authors in [3], Theorem 1, have proved that there exist a constant  $h_0 \in (0, 1)$  such that (3.2) holds for all  $|t-s| \leq h_0$  and  $|x-y| \leq h_0$ . There  $h_0$  is not allowed to be larger than 1. In our case, there is no restriction on  $h_0$ .

#### 4. Optimality of Hölder Exponents

In this Section, we investigate the optimality of the estimate (3.2) for

$$u(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-\theta}(x-\eta) W(d\theta, d\eta), \quad t, x \in [0, T] \times \mathbb{R}.$$

For the optimality of the Hölder exponents of a random field which is the solution of a stochastic partial differential equation, we are only aware of Theorem 5.1 in [12].

Our main purpose in this Section is to prove

**Theorem 4.1.** *Assume that  $u_0 = v_0 = 0$ ,  $b = 0$  and  $\sigma = I_{d \times d}$ .*

1. *Fix  $t \in [0, T]$  and a compact interval  $J$ . There is a constant  $C > 0$  such that for all  $x, y \in J$ ,*

$$\|u(t, x) - u(t, y)\|_{L^2} \geq C|x - y|^H. \quad (4.1)$$

*Consequently, a.s., the mapping  $x \mapsto u(t, x)$  is not in  $\mathcal{C}_{H+}(\mathbb{R})$ .*

2. *Fix  $x \in \mathbb{R}$  and  $t_0 \in (0, T]$ . There is a constant  $C > 0$  such that for all  $t, s \in [t_0, T]$  with  $|t - s|$  sufficiently small,*

$$\|u(t, x) - u(s, x)\|_{L^2} \geq \begin{cases} C|t - s|^{\frac{H}{2}}, & \text{for SHE;} \\ C|t - s|^H, & \text{for SWE.} \end{cases} \quad (4.2)$$

*Consequently, a.s., the mapping  $t \mapsto u(t, x)$  is not in  $\mathcal{C}_{\frac{H}{2}+}(\mathbb{R}_+)$  for SHE or  $\mathcal{C}_{H+}(\mathbb{R}_+)$  for SWE.*

*Proof of Theorem 4.1.* For any  $x \in \mathbb{R}$ , set  $R(x) := \mathbb{E}[u(t, 0)u(t, x)]$ . It is clear that  $\mathbb{E}[|u(t, x) - u(t, y)|^2] = 2(R(0) - R(x - y))$ , and then to prove (4.1) it suffices to show that for any  $x \in \mathbb{R}$ , there exist  $C > 0$  such that

$$R(0) - R(x) \geq C|x|^H.$$

Without loss of generality, we assume that  $t = 1$ .

By Itô isometry (2.9) and simple calculations, we have

$$\begin{aligned} R(0) - R(x) &= c_H \int_0^t \int_{\mathbb{R}} \left( 1 - \frac{e^{i\xi x} + e^{-i\xi x}}{2} \right) |\mathcal{F}G_{t-s}(\xi)|^2 |\xi|^{1-2H} d\xi ds \\ &= \begin{cases} c_H \int_{\mathbb{R}} \frac{1 - \cos(\xi x)}{|\xi|^{2H+1}} (1 - e^{-|\xi|^2}) d\xi, & \text{for SHE;} \\ \frac{1}{2} c_H \int_{\mathbb{R}} \frac{1 - \cos(\xi x)}{|\xi|^{2H+1}} \left( 1 - \frac{\sin(2|\xi|)}{2|\xi|} \right) d\xi, & \text{for SWE.} \end{cases} \end{aligned}$$

The integrands above are both non-negative, and for  $|\xi| > 1$ ,

$$1 - e^{-\frac{|\xi|^2}{2}} > 1 - e^{-\frac{1}{2}}, \quad 1 - \frac{\sin(2|\xi|)}{2|\xi|} \geq \frac{1}{2}.$$

Applying change of variables  $y = x\xi$ , we obtain

$$R(0) - R(x) \geq C|x|^{2H} \int_{|y| \geq |x|} \frac{1 - \cos y}{|y|^{2H+1}} dy.$$

Since  $x \in J$  with  $J$  compact, the last integral is bounded below by a positive constant. This proves (4.1).

We now turn to proof of (4.2) for  $t_0 \leq s < t \leq T$ . In this situation, by Itô isometry (2.9),

$$\begin{aligned} & \mathbb{E} \left[ |u(t, x) - u(s, x)|^2 \right] \\ &= \mathbb{E} \left[ \left| \int_s^t \int_{\mathbb{R}} G_{t-\theta}(x-\eta) W(d\theta, d\eta) \right|^2 \right] + \mathbb{E} \left[ \left| \int_0^s \int_{\mathbb{R}} (G_{t-\theta}(x-\eta) - G_{s-\theta}(x-\eta)) W(d\theta, d\eta) \right|^2 \right] \\ &= C_H \int_s^t \int_{\mathbb{R}} |\mathcal{F}G_{t-\theta}(\xi)|^2 |\xi|^{1-2H} d\theta d\xi + C_H \int_0^s \int_{\mathbb{R}} |\mathcal{F}G_{t-\theta}(\xi) - \mathcal{F}G_{s-\theta}(\xi)|^2 |\xi|^{1-2H} d\theta d\xi \\ &=: C_H I_1 + C_H I_2. \end{aligned}$$

Applying (2.4), we obtain an estimate of  $I_1$ :

$$I_1 = \begin{cases} C_1(t-s)^H, & \text{for SHE;} \\ C_2(t-s)^{2H+1}, & \text{for SWE.} \end{cases}$$

For  $I_2$ , we estimate separately for SHE and SWE.

For SHE, simple calculations yield

$$\int_0^s |\mathcal{F}G_{t-\theta}(\xi) - \mathcal{F}G_{s-\theta}(\xi)|^2 d\theta = \left( 1 - \exp\left(-\frac{(t-s)|\xi|^2}{2}\right) \right)^2 \frac{1 - \exp(-s|\xi|^2)}{|\xi|^2}.$$

Then by change of variables  $y = \sqrt{t-s}\xi$ , we obtain

$$\begin{aligned} I_2 &\geq \int_{|\xi| \geq \frac{1}{t-s}} \left( 1 - \exp\left(-\frac{(t-s)|\xi|^2}{2}\right) \right)^2 \frac{1 - \exp(-s|\xi|^2)}{|\xi|^{2H+1}} d\xi \\ &\geq \left( 1 - \exp\left(-\frac{t_0}{t-s}\right) \right) \int_{|\xi| \geq \frac{1}{t-s}} \left( 1 - \exp\left(-\frac{(t-s)|\xi|^2}{2}\right) \right)^2 \frac{1}{|\xi|^{2H+1}} d\xi \\ &= (t-s)^H \left( 1 - \exp\left(-\frac{t_0}{t-s}\right) \right) \int_{|y| \geq 1} \left( 1 - \exp\left(-\frac{|y|^2}{2}\right) \right)^2 \frac{1}{|y|^{2H+1}} dy \\ &\geq (t-s)^H \left( 1 - \exp\left(-\frac{t_0}{t-s}\right) \right) \frac{(\sqrt{e}-1)^2}{eH}. \end{aligned}$$

For sufficient close  $s$  and  $t$ , say  $t-s \leq \frac{t_0}{\ln 2}$ , one has  $1 - \exp(-\frac{t_0}{t-s}) \geq \frac{1}{2}$ , and then

$$I_2 \geq \frac{(\sqrt{e}-1)^2}{2eH} (t-s)^H.$$

For SWE, we have

$$\begin{aligned} & \int_0^s |\mathcal{F}G_{t-\theta}(\xi) - \mathcal{F}G_{s-\theta}(\xi)|^2 d\theta \\ &= \frac{t(1-\cos((t-s)|\xi|))}{|\xi|^2} + \frac{(1-\cos((t-s)|\xi|)) \sin((t+s)|\xi|)}{2|\xi|^3} + \frac{\sin(2(t-s)|\xi|) + 2\sin((t-s)|\xi|)}{4|\xi|^3}. \end{aligned}$$

Then by using change of variables  $y = (t-s)\xi$ , we obtain

$$\begin{aligned} I_2 &\geq t_0(t-s)^{2H} \int_{|y| \geq 1} \frac{1-\cos(|y|)}{|y|^{2H+1}} dy - \frac{(t-s)^{2H+1}}{2} \int_{|y| \geq 1} \frac{1}{|y|^{2H+1}} dy - \frac{3(t-s)^{2H+1}}{4} \int_{|y| \geq 1} \frac{1}{|y|^{2+2H}} dy \\ &= C_1(t-s)^{2H} - \frac{5H+1}{2H(2H+1)} (t-s)^{2H+1}, \end{aligned}$$

where  $C_1 = t_0 \int_{|y| \geq 1} \frac{1 - \cos(|y|)}{|y|^{2H+1}} dy$  is positive and bounded. Then when  $t - s \leq \frac{H(2H+1)C_1}{5H+1}$ , one has

$$I_2 \geq \frac{C_1}{2}(t - s)^{2H}.$$

This proves (4.2).

To concern the absence of Hölder continuity, we need the following Fernique-type theorem which says that an a.s. uniformly bounded centered, Gaussian process has bounded moments.

**Lemma 4.1.** *For  $X$  centered, Gaussian,*

$$\mathbb{P} \left\{ \sup_{t \in \mathcal{T}} X(t) < \infty \right\} = 1 \iff \mathbb{E} \left[ \exp \left( \alpha \left| \sup_{t \in \mathcal{T}} X(t) \right|^2 \right) \right] < \infty \quad (4.3)$$

for sufficiently small  $\alpha > 0$ .

*Proof of Lemma 4.1.* See [1], Theorem 3.2. □

Now we consider the absence of Hölder continuity of  $u$  through Lemma 4.1. It is clear that  $u(t, x) - u(s, x)$  and  $u(t, x) - u(t, y)$  are both centered, Gaussian for any  $0 \leq s < t \leq T$  and  $x, y \in \mathbb{R}$ . We only give details for the space variable, while the arguments are available for the time variable.

Suppose that for a fixed  $t \in (0, T]$ , the sample paths  $x \mapsto u(t, x)$  are  $\gamma$ -Hölder continuous for some  $\gamma > H$ . Then for any compact interval  $J$ , there exists  $C(\omega) \in (0, \infty)$  such that

$$\sup_{x, y \in J, x \neq y} \frac{|u(t, x) - u(t, y)|}{|x - y|^\gamma} < C(\omega).$$

This yields that the centered Gaussian process

$$\left\{ \frac{u(t, x) - u(t, y)}{|x - y|^\gamma}, \quad x, y \in J, x \neq y \right\}$$

is finite a.s., from which we conclude by Lemma 4.1 that

$$\mathbb{E} \left[ \sup_{x, y \in J, x \neq y} \left| \frac{u(t, x) - u(t, y)}{|x - y|^\gamma} \right|^2 \right] < \infty.$$

In particular, there would exist a finite  $C > 0$  such that

$$\mathbb{E}[|u(t, x) - u(t, y)|^2] \leq C|x - y|^\gamma,$$

which contracts (4.1). □

Now we can prove Theorem 1.1.

*Proof of Theorem 1.1.* Theorem 1.1 follows from Theorem 3.1 and 4.1. □

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